Cyclostationary measurement of low-frequency odd moments of current fluctuations

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(Received 10 March 2004; revised manuscript received 27 September 2004; published 16 February 2005)

Measurement of odd moments of current fluctuations is difficult due to strict requirements for band-pass filtering. We propose how these requirements can be overcome using cyclostationary driving of the measured signal and indicate how the measurement accuracy can be tested through the phase dependence of the moments of the fluctuations. We consider two schemes, the mixing scheme and the statistics scheme, where the current statistics can be accessed. We also address the limitations of the schemes, due to excess noise and due to the effects of the environment, and, finally, discuss the required measurement times for typical setups.

I. INTRODUCTION

The discrete nature of the charge carriers shows up in the peculiar statistics of the transmitted current through mesoscopic systems.[1] Consequently, these statistics in general cannot be described through the usual Gaussian probability density which is completely determined by its first two moments (cumulants), average current and noise, whereas its higher cumulants vanish (for a difference between the moments and cumulants, see Ref. 2). Recently, there have been many theoretical predictions for the behavior of either the third moment of fluctuations or the full counting statistics in different types of mesoscopic systems (see a review in Ref. 3), including normal-metal tunnel junctions,[4] normal-metal—superconductor[5,6] and superconductor-superconductor junctions,[7–9] diffusive[10–13] and chaotic wires,[14] double-barrier junctions[15] and Coulomb-blockaded systems.[16] For example, the second and third moments of current fluctuations through a phase-coherent scatterer at $T=0$ may be expressed as

$$S \equiv \langle \delta I \delta I \rangle = \frac{e^3 |V|}{h} \sum_n T_n (1 - T_n) \equiv F_2 e |I|,$$

$$K \equiv \langle \delta I \delta I \delta I \rangle = \frac{e^4 V}{h} \sum_n T_n (1 - T_n)(1 - 2T_n) \equiv F_3 e^2 I, \tag{1}$$

where T_n are transmission eigenvalues. Their distribution depends on the properties of the scatterer,[17] and it can be characterized by defining the Fano factors F_2 and F_3. Compared to the average current I, the higher moments may thus reveal additional information about the scatterer (for an example, see Ref. 18). Moreover, $K \neq 0$ implies that the distribution of fluctuating currents is "skew." This has important consequences in the situations where the driven current fluctuations act as an environment to another mesoscopic system.[19,20]

However, up to date there exists only one published measurement of the higher (than second) moments.[18,21] One of the reasons for this is the necessary conditions for the filtering in these devices: Impedance matching between the amplifier and the sample is possible typically only within a fairly narrow frequency band, outside of which the signal

from the sample does not couple to the measurement device. However, as we show below, for the typical measurements of the odd nth moments of fluctuations, the requirement for the bandwidth $2\delta\omega$ of the measured frequencies around the mean frequency ω_0 is $\delta\omega > \omega_0/n$.[21] Especially for the measurement of high-impedance samples (Z larger than 1 kΩ) this requirement is very hard if not impossible. In this paper, we show how this requirement can be circumvented by a cyclostationary driving of the measured system (see Fig. 1).

To motivate the use of cyclostationary driving, consider the n'th central moment of current fluctuations in the frequency space

$$\widetilde{M}^{(n)}(\omega_1, \omega_2, \ldots, \omega_n) \equiv \langle \delta I(\omega_1) \delta I(\omega_2) \ldots \delta I(\omega_n) \rangle, \tag{2}$$

where the brackets denote ensemble averaging and $\delta I(\omega) \equiv I(\omega) - \langle I(\omega) \rangle$. If the studied system is time independent, i.e., if it is driven with a constant voltage, we get the signal only from the frequencies that satisfy

$$\omega_1 + \omega_2 + \cdots + \omega_n = 0. \tag{3}$$

Consider now a situation where the signal is band-pass filtered before its correlators are measured, such that only the frequencies in the band $\omega \in [\omega_0 - \delta\omega, \omega_0 + \delta\omega]$ pass through the filters. If $\delta\omega \ll \omega_0$, we cannot fulfill condition (3) for odd n. Simple algebra shows that for the nth moment (where n is odd), we only get some signal if

$$\delta\omega > \omega_0/n. \tag{4}$$

However, this condition is true for *stationary* signal, but if we adiabatically drive the system with frequency ω_D, condition (3) changes to

$$\omega_1 + \omega_2 + \ldots + \omega_n = m\omega_D, \tag{5}$$

with an integer m. Reasonable examples considering the odd nth moments are $\omega_D = \omega_0, 3\omega_0, \ldots, n\omega_0$.

In the second section of this paper, we detail two specific measurement schemes using cyclostationary driving and compare them to that applied in Ref. 21. The first of the schemes is based on mixing the signal with itself and with an oscillatory function $f(t)$, and the second on measuring the statistics of the output signal from the sample after filtering and mixing with $f(t)$. The third section discusses the effect of
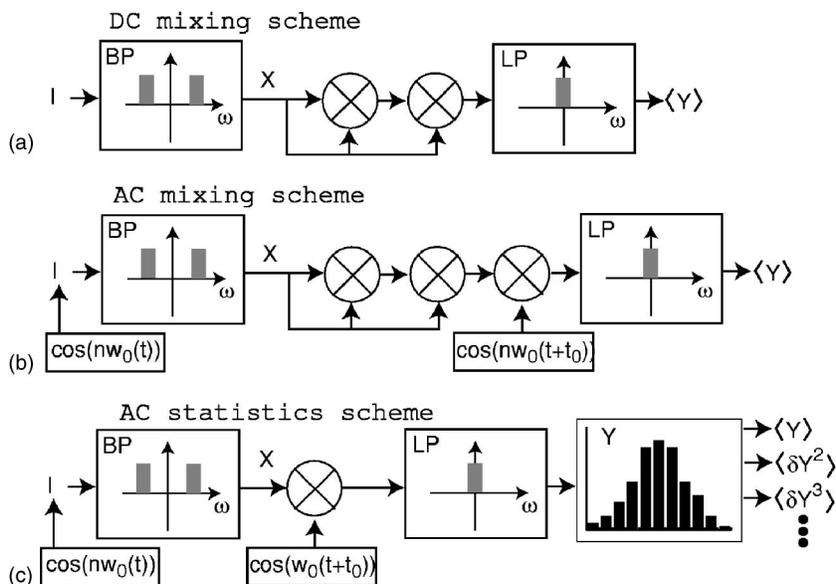
FIG. 1. Measurement schemes considered in this paper. Each contain an input I filtered through a band-pass filter (denoted with BP, corresponding to the function H_{BP}). In the DC mixing scheme, analogous to that utilized in Ref. 21, the stationarily driven and band-pass filtered input is mixed twice with itself, low-pass filtered (denoted with LP, corresponding to the function H_{LP}^m) and averaged. The "AC mixing scheme" is a modification of the DC scheme, including a cyclostationary driving of the measured system, and mixing with an extra function $f(t) = \cos(n\omega_0(t+t_0))$, synchronized with the driving and with a controlled phase shift t_0. In the "AC statistics scheme," the statistics of the output is measured and the driven and filtered signal is mixed only with $f(t)$.

cyclostationary driving in some example systems and the fourth considers the limitations of the presented measurement schemes due to the excess noise in the environment, and estimates the required averaging time for obtaining an accurate signal. The details of the calculations are presented in the Appendix.

II. MEASUREMENT SCHEMES

Let us consider the specific measurement schemes for measuring the moments of current fluctuations depicted in Fig. 1. The emphasis of this analysis is on the third moment, but the results and the schemes are fairly straightforward to generalize also to higher moments. In both of these schemes, the studied system is driven with a time-dependent voltage of the form (for most of the discussion below, the different harmonics do not need to be in the same phase, but we assume them in phase for simplicity)

$$V(t) = V_0 + V_1 \cos(\omega_0 t) + V_3 \cos(3\omega_0 t). \qquad (6)$$

This voltage produces in general an output current of the form

$$I(t) = \sum_l I_l \cos(l\omega_0 t). \qquad (7)$$

In linear systems with resistance R, $I_l = V_l/R$ and thus $I_l = 0$ for $l > 3$. In nonlinear systems, the output current would also contain higher harmonics of ω_0, but these do not contribute essentially to the results. However, the coefficients I_l should be taken from the proper Fourier analysis of the output current.

With pure voltage driving, the moments of the low-frequency current fluctuations are proportional to either the average current $I(t)$ (for odd moments) or its absolute value (for even moments). This holds as long as the driving is adiabatic, i.e., the highest driving frequency $3\omega_0$ is low compared to the internal energy scales of the probed system or to $eV(t)/\hbar$.[12,22] In fact, any slow measurement of the signal as a function of the driving voltage can be considered as "adiabatic" in this sense—the only difference to that considered here is that typically the timescales of such voltage variations are of the order of seconds, much slower than the scales for band-pass filtering. Furthermore, for low frequencies, the fluctuations can be considered "white," i.e., they are of the form (below, we reserve the symbol S for the second moment, i.e., noise, and use the symbol K for the third moment)[23]

$$S(t,t') \equiv \langle \delta I(t) \delta I(t') \rangle = \sum_l S_l \cos(l\omega_0 t) \delta(t-t') \qquad (8)$$

and

$$K(t,t',t'') \equiv \langle \delta I(t) \delta I(t') \delta I(t'') \rangle$$
$$= \sum_l K_l \cos(l\omega_0 t) \delta(t-t') \delta(t-t''). \qquad (9)$$

Similarly, the nth central moment would be

$$M^{(n)}(t_1, t_2, \ldots, t_n) \equiv \langle \delta I(t_1) \delta I(t_2) \ldots \delta I(t_n) \rangle$$
$$= \sum_l M_l^{(n)} \cos(l\omega_0 t_1) \delta(t_1 - t_2) \delta(t_1 - t_3) \ldots$$
$$\delta(t_1 - t_n). \qquad (10)$$

In most cases below, it is enough to cut the series in the nth harmonic (for the nth moment) as the higher ones do not

contribute to the results. There are many sources for the higher harmonics than those present in the driving and one has to always first do the proper Fourier analysis to find these, prior to examining the results: (i) finite temperature of the order of the voltage V makes the relations $S(V)$ and $K(V)$ nonlinear even in many linear systems,[13,24] (ii) if the sign of the total current is allowed to vary, one has to take the Fourier components from the absolute value of the current for $S(V)$, and (iii) these harmonics arise naturally in nonlinear systems. Examples of these are discussed in Sec. III. However, we stress that all these effects can only make rise to frequency components of the form $l\omega_0$, where l is some integer.

For what follows, the assumption of the "white" fluctuations is in fact less strict than assumed in Eqs. (8)–(10): it is enough that the fluctuations are frequency independent in the window allowed by the band-pass filter.

For both schemes, the signal is assumed to be filtered with an ideal band-pass filter allowing frequencies $\omega \in [\omega_0 - \delta\omega, \omega_0 + \delta\omega]$. Such a filter can be described through the boxcar function $\widetilde{H}_{BP}(\omega) \equiv \theta(|\omega| - (\omega_0 - \delta\omega)) - \theta(|\omega| - (\omega_0 + \delta\omega))$, where $\theta(\omega)$ is the Heaviside step function. Thus, the outgoing signal from the filter is in the time domain a convolution (below, unless it is obvious, the functions in the frequency domain are denoted with a tilde)

$$X(t) = \frac{1}{\sqrt{2\pi}} H_{BP}(t) * I(t). \quad (11)$$

At the end of the processing, the signal is gathered for a time exceeding $1/\omega_0$. Such a process can be described by a convolution with a low-pass filter $\widetilde{H}_{LP}^m(\omega) = 1 - \theta(|\omega| - \omega_m)$, where $\omega_m \ll \omega_0$.

A. Mixing schemes

In the mixing schemes depicted in Fig. 1 [schemes (a) and (b)], we assume that the signal is, after filtering, twice mixed with itself and with a function $f(t)$ specified below, low-pass filtered, and finally ensemble averaged. Thus, the measurement result is

$$\langle Y \rangle \equiv \frac{1}{\sqrt{2\pi}} \langle H_{LP}^m * [f(t) X^3(t)] \rangle. \quad (12)$$

Assuming that over a sequence of measurements performed for the averaging, the filters and the function $f(t)$ do not essentially change, one obtains

$$\langle Y \rangle = \frac{1}{4\pi^2} H_{LP}^m * [f(t') \langle (H_{BP} * I)^3 \rangle]$$

$$= \frac{1}{4\pi^2} H_{LP}^m * \left[f(t') \int dt_1 dt_2 dt_3 H_{BP}(t' - t_1) \right.$$

$$\left. \times H_{BP}(t' - t_2) H_{BP}(t' - t_3) \langle I(t_1) I(t_2) I(t_3) \rangle \right]. \quad (13)$$

Below, we take a closer look at two ways of driving the signal: first with a DC bias and wide band-pass filters, and then with an AC bias and narrow band-pass filters.

1. DC mixing scheme

Assume first that the system is DC biased, i.e., only the first term in Eqs. (6)–(9) is nonzero. In this case, we do not need the function generator, i.e., $f(t) = 1$. Such a setup corresponds to the one utilized in Ref. 21 and it is depicted in Fig. 1(a).

One can write the current in the form $I(t) = \langle I \rangle + \delta I(t)$, where $\delta I(t)$ is a zero-mean fluctuating current. Then the third raw moment of current fluctuations is

$$\langle I(t) I(t') I(t'') \rangle = \langle I \rangle^3 + \langle I \rangle S[\delta(t - t') + \delta(t - t'') + \delta(t' - t'')]$$

$$+ K\delta(t - t') \delta(t - t''). \quad (14)$$

The average signal from the sampling filter H_{LP}^m is

$$\langle Y \rangle = \frac{1}{4\pi^2} \langle H_{LP}^m * [H_{BP} * I(t)]^3 \rangle$$

$$= \frac{1}{4\pi^2} H_{LP}^m * \int dt_1'' dt_2'' dt_3'' H_{BP}(t' - t_1'') H_{BP}(t' - t_2'')$$

$$\times H_{BP}(t' - t_3'') \langle I(t_1'') I(t_2'') I(t_3'') \rangle$$

$$= \frac{1}{4\pi^2} H_{LP}^m * \left[\left(\langle I \rangle \int dt'' H_{BP}(t' - t'') \right)^3 + 3\langle I \rangle S \right.$$

$$\times \left(\int dt'' H_{BP}(t' - t'') \right) \left(\int dt'' H_{BP}^2(t' - t'') \right)$$

$$\left. + K \int dt'' H_{BP}^3(t' - t'') \right]. \quad (15)$$

The band-pass filter filters out the DC signal,

$$\int dt'' H_{BP}(t - t'') = \sqrt{2\pi} \widetilde{H}_{BP}(0) = 0, \quad (16)$$

as long as it is a band-pass filter, i.e., $\delta\omega < \omega_0$. Therefore, only the last term remains, and we obtain

$$\langle Y \rangle = K \frac{3}{4\pi^2} (3\delta\omega - \omega_0)^2 \theta(3\delta\omega - \omega_0). \quad (17)$$

It can be seen that for $\delta\omega < \omega_0/3$, $\langle Y \rangle = 0$, in agreement with the discussion in Sec. I. The scaling of the observed quantity as a function of the bandwidth was applied in Ref. 21 to show that it was indeed the third moment that was measured.

2. AC mixing scheme

Assume that the current, noise and the third moment are of the form given in Eqs. (7), (8), and (9) respectively. For simplicity, we assume that the lowest driving frequency equals the center frequency of the band-pass filter and denote both by ω_0. The average output signal is given by

$$\langle Y \rangle = \frac{1}{4\pi^2} H_{LP}^m * \left\{ f(t') \left[\left(\sum_l I_l \int dt'' H_{BP}(t' - t'') \cos(l\omega_0 t'') \right)^3 \right. \right.$$

$$+ 3 \left(\sum_l I_l \int dt'' H_{BP}(t' - t'') \cos(l\omega_0 t'') \right)$$

$$\times \left(\sum_l S_l \int dt'' H_{BP}^2(t' - t'') \cos(l\omega_0 t'') \right)$$

$$\left. \left. + \sum_l K_l \int dt'' H_{BP}^3(t' - t'') \cos(l\omega_0 t'') \right] \right\}. \quad (18)$$

Now we have three types of integrals with the band-pass filter H_{BP}. We obtain

$$\int dt'' H_{BP}(t' - t'') \cos(l\omega_0 t'') = \sqrt{2\pi} \widetilde{H}_{BP}(l\omega_0) \cos(l\omega_0 t')$$

$$= \sqrt{2\pi} \delta_{l,1} \cos(\omega_0 t'), \quad (19)$$

$$\int dt'' H_{BP}^2(t' - t'') \cos(l\omega_0 t'') = (\widetilde{H}_{BP} * \widetilde{H}_{BP})|_{\omega=l\omega_0} \cos(l\omega_0 t')$$

$$= 2\delta\omega[2\delta_{l,0} + \delta_{l,2} \cos(2\omega_0 t')], \quad (20)$$

$$\int dt'' H_{BP}^3(t' - t'') \cos(l\omega_0 t'')$$

$$= \frac{1}{\sqrt{2\pi}} (\widetilde{H}_{BP} * \widetilde{H}_{BP} * \widetilde{H}_{BP})|_{\omega=l\omega_0} \cos(l\omega_0 t')$$

$$= \frac{3\delta\omega^2}{\sqrt{2\pi}} [3\delta_{l,1} \cos(\omega_0 t') + \delta_{l,3} \cos(3\omega_0 t')]. \quad (21)$$

Combining these results, we get

$$\langle Y \rangle = \frac{H_{LP}^m}{\sqrt{8\pi^3}} * \left\{ f(t') \left[2\pi I_1^3 \cos^3(\omega_0 t') + 6I_1 \delta\omega[2S_0 \cos(\omega_0 t') \right. \right.$$

$$+ S_2 \cos(2\omega_0 t')] + \frac{3\delta\omega^2}{2\pi} [3K_1 \cos(\omega_0 t')$$

$$\left. \left. + K_3 \cos(3\omega_0 t')] \right] \right\}. \quad (22)$$

Now it is time to specify the mixing function $f(t)$. This mixes some of the above oscillating functions to the zero frequency and thereafter the low-pass filter H_{LP}^m filters the other frequencies out. There are two meaningful choices for $f(t)$, (a) $f(t) = \cos(\omega_0(t+t_0))$ and (b) $f(t) = \cos(3\omega_0(t+t_0))$, where t_0 represents the phase shift between the function generator and the signal. With the first choice, the result is

$$\langle Y_a \rangle = \left[\frac{3}{8} I_1^3 + 6 \frac{\delta\omega}{2\pi} I_1 S_0 + \frac{9\delta\omega^2}{8\pi^2} K_1 \right] \cos(\omega_0 t_0), \quad (23)$$

and with the second,

$$\langle Y_b \rangle = \left(\frac{1}{8} I_1^3 + \frac{3\delta\omega^2}{8\pi^2} K_3 \right) \cos(3\omega_0 t_0). \quad (24)$$

This result illustrates that, in a linear system, setting $I_0 = I_1 = I_2 = 0$, the signal with $f(t) = \cos(3\omega_0(t+t_0))$ is dependent only on the third moment K_3. Moreover, the dependence on the phase shift shows that the function $f(t)$ and the driving have to be synchronized. This phase dependence can also be utilized as a signature of the fact that the measured quantity really corresponds to the properties of the signal coming from the sample, analogously to the dependence on the bandwidths utilized in Ref. 21.

B. AC statistics scheme

In the second scheme [Fig. 1(c)], the AC-driven current is mixed only once with the function $f(t) = \cos(\omega_0(t+t_0))$ and the output signal

$$Y = \frac{1}{2\pi} H_{LP}^m * [f(H_{BP} * I)] \quad (25)$$

is measured many times. This produces an ensemble of Y-values, whose moments are related to the moments of the current $I(t)$. As above, we assume that the bandwidths of both the band-pass filter, $\delta\omega$, and of the sampling, ω_m, are much lower than ω_0.

The nth central moment of this Y-distribution is given by

$$\langle (\delta Y)^n \rangle = \sum_{l=0}^{\infty} \frac{1}{(2\pi)^n}$$

$$\times \int \left[\prod_{i=1}^{n} dt_i H_{LP}^m(t_i) f(t_i) \right] \int \left[\prod_{i=1}^{n} dt_i' H_{BP}(t_i - t_i') \right]$$

$$\times \left\langle \prod_{i=1}^{n} \delta I(t_i') \right\rangle_l, \quad (26)$$

where l labels harmonics of the base frequency ω_0. Utilizing the white-noise assumption as in Eqs. (8)–(10) yields

$$\langle (\delta Y)^n \rangle = \sum_{l=0}^{\infty} \frac{1}{(2\pi)^n} \int \left[\prod_{i=1}^{n} dt_i H_{LP}^m(t_i) f(t_i) \right]$$

$$\times \int dt' \left[\prod_{i=1}^{n} H_{BP}(t_i - t') \right] M_l^{(n)} \cos(l\omega_0 t'), \quad (27)$$

where $M_l^{(n)}$ is the coefficient of the lth harmonic of the nth central moment in the measured signal. This integral is evaluated in Appendix A. The results for the six lowest moments are (the first, average, is the raw moment, the rest are central moments, i.e., defined, with respect to the average)

$$\langle Y \rangle = \frac{1}{2} I_1 \cos(\omega_0 t_0), \quad (28)$$

$$\langle (\delta Y)^2 \rangle = \frac{\omega_d}{4\pi} [2S_0 + S_2 \cos(2\omega_0 t_0)], \quad (29)$$

$$\langle(\delta Y)^3\rangle = \frac{3\omega_d^2}{32\pi^2}[3K_1\cos(\omega_0 t_0) + K_3\cos(3\omega_0 t_0)], \quad (30)$$

$$\langle(\delta Y)^4\rangle = \frac{\omega_d^3}{24\pi^3}\left[6M_0^{(4)} + 4M_2^{(4)}\cos(2\omega_0 t_0) + M_4^{(4)}\cos(4\omega_0 t_0)\right],$$
$$(31)$$

$$\langle(\delta Y)^5\rangle = \frac{115\omega_d^4}{6144\pi^4}\left[10M_1^{(5)}\cos(\omega_0 t_0) + 5M_3^{(5)}\cos(3\omega_0 t_0)\right.$$
$$\left. + M_5^{(5)}\cos(5\omega_0 t_0)\right], \quad (32)$$

$$\langle(\delta Y)^6\rangle = \frac{11\omega_d^5}{1280\pi^5}\left[20M_0^{(6)} + 15M_2^{(6)}\cos(2\omega_0 t_0)\right.$$
$$\left. + 6M_4^{(6)}\cos(4\omega_0 t_0) + M_6^{(6)}\cos(6\omega_0 t_0)\right], \quad (33)$$

where $\omega_d = \min(\omega_m, \delta\omega)$. Note that which harmonics are nonzero depends on how the system is driven. For example, if one drives the system with a voltage $V_3\cos(3\omega_0 t)$, the only finite moments (among the first six) contributing to the measured signal are S_0, K_3, $M_0^{(4)}$, $M_3^{(5)}$, $M_0^{(6)}$, and $M_6^{(6)}$. Such examples are discussed in more detail in the following section.

III. EXAMPLES OF HIGHER HARMONICS

In this section, we discuss examples of systems where one could measure the third moment of fluctuations with AC driving. The considered examples are the simplest linear systems at low temperatures and the finite-temperature case where part of the signal is mixed to higher harmonics. If the nonlinearities are relevant, one may consider making a differential measurement similar to the lock-in schemes typically applied for measuring the differential conductance: on top of a DC driving voltage, one adds a low-amplitude AC bias, which oscillates with the relevant frequency.[25] Assuming the amplitude of the AC bias is low enough, the oscillating part of the noise is directly proportional to it, and the proportionality depends only on the Fano factor at that bias point. However, such a scheme has the disadvantage that the second moment of the outcome signal in the AC statistics scheme is proportional to the total average current, while the probed third moment is dependent only on the small oscillating bias. This makes the averaging time of the signal quite long; see Sec. IV C. Therefore, below we mostly consider cases without the DC bias.

A. Linear systems at low temperatures

The simplest but yet important example of a system where our theory is applicable is a linear system with resistance Z_s, driven with an AC voltage $V(t) = I_3 Z_s \cos(3\omega_0 t)$. In a tunnel junction, neglecting the effects of its environment,[21,26] the third moment is independent of the temperature[4] whereas the noise is in general a mixture of shot noise (produced by driving) and thermal noise.[24] In other systems, also the third moment becomes temperature

dependent.[13] For $eV \gg k_B T$, the output current, noise and the third moment are

$$I(t) = I_3\cos(3\omega_0 t), \quad (34)$$

$$S(t) = eF_2 I_3|\cos(3\omega_0 t)| = eF_2 I_3\left(\frac{2}{\pi} + \frac{4}{3\pi}\cos(6\omega_0 t) + \cdots\right), \quad (35)$$

$$K(t) = e^2 F_3 I_3\cos(3\omega_0 t). \quad (36)$$

Here F_2 and F_3 are the Fano factors ($F_2 = F_3 = 1$ for a tunnel junction) for the second and third moments, respectively. Applying the AC mixing scheme and mixing with $f(t) = \cos(3\omega_0(t+t_0))$ yields the output

$$\langle Y\rangle = \frac{3}{2}\left(\frac{\delta\omega}{2\pi}\right)^2 e^2 F_3 I_3. \quad (37)$$

In the AC statistics scheme, the average of the output signal vanishes, the variance is

$$\langle(\delta Y)^2\rangle = \frac{\min(\omega_m, \delta\omega)}{\pi^2} eF_2 I_3 \quad (38)$$

and the third moment is

$$\langle(\delta Y)^3\rangle = \frac{3}{8}\left(\frac{\min(\omega_m, \delta\omega)}{2\pi}\right)^2\cos(3\omega_0 t_0)e^2 F_3 I_3. \quad (39)$$

This illustrates that, as long as the thermal noise and environmental noise are negligible, the best signal from the third moment (compared to the second) is obtained with wideband filtering but low-amplitude driving.

For a specific example, assume a realistic set of numbers, $I_3 = 100$ nA and $\min(\omega_m, \delta\omega)/(2\pi) = 10$ MHz. Then, we get

$$\langle Y^2\rangle \approx F_2(320\text{ pA})^2, \quad (40)$$

$$\langle Y^3\rangle \approx F_3(46\text{ pA})^3\cos(3\omega_0 t_0). \quad (41)$$

Thus, the third moment should be well observable in this situation. With larger currents or smaller bandwidths, the second moment becomes relatively larger, but only as $(\langle Y^2\rangle)^{1/2}/(\langle Y^3\rangle)^{1/3} \propto (I_3/(e\delta\omega))^{1/6}$.

B. Diffusive wire at a finite temperature

The temperature dependence of the higher moments of fluctuations depends on the studied system. For example, for a diffusive wire one expects[13]

$$S = \frac{eI}{3}\coth\left(\frac{eV}{2k_B T}\right) + \frac{4k_B T}{3Z_s}, \quad (42)$$

$$K = e^2 I\frac{6(-1+e^{4p}) + (1-26e^{2p}+e^{4p})p}{15p(-1+e^{2p})^2}. \quad (43)$$

Here $p = eV/2k_B T$. In this case, for an oscillating voltage, some of the moments mix into higher frequencies, altering the expected outcome of the measurement. However, in the
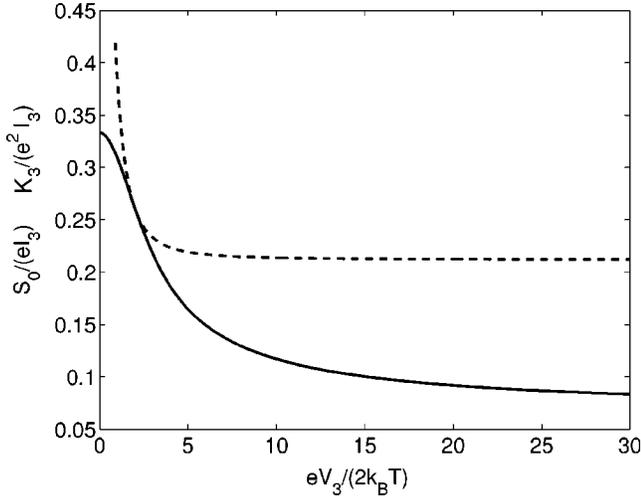
FIG. 2. Stationary part $S_0 - 4k_BT/(3Z_s)$ of the second moment (dashed line) and lowest harmonic K_3 of the third moment (solid line) in a diffusive wire driven with $V_3 \cos(3\omega_0 t)$, normalized to eI_3 and $e^2 I_3$, respectively. Note that the previous diverges with $eV_3 \ll k_BT$ as then S_0 contains only the thermal noise and is essentially independent of I_3.

extremal cases $eV_i \ll k_BT$ and $eV_i \gg k_BT$, where eV_i is the amplitude of the oscillating voltage, this mixing is not important and one may simply use

$$S(eV_i \ll k_BT) = \frac{2k_BT}{Z_s}, \qquad (44)$$

$$S(eV_i \gg k_BT) = \frac{1}{3}e|I_i(t)|, \qquad (45)$$

$$K(eV_i \ll k_BT) = \frac{1}{3}e^2 V_i(t), \qquad (46)$$

$$K(eV_i \gg k_BT) = \frac{1}{15}e^2 V_i(t). \qquad (47)$$

Here Z_s is the resistance of the measured sample.

For example, let us assume that we drive the system with the voltage

$$V(t) = V_3 \cos(3\omega_0 t). \qquad (48)$$

As a diffusive wire is a linear system, the average current follows the oscillations of the voltage, $I(t) = I_3 \cos(3\omega_0 t)$. Our measuring schemes yield information about the stationary part S_0 of the noise, and about the $3\omega_0$-component K_3 of the third moment. These are plotted in Fig. 2 as a function of $eV_3/2k_BT$. One can observe that in this particular case, the crossover to the "pure shot noise" takes place for $eV_3 \gtrsim 10k_BT$ whereas the crossover between the low- and high-temperature third moments is wider, saturating only at some $eV_3 \gtrsim 40k_BT$.

IV. LIMITATIONS OF MEASUREMENT

In this section, we discuss the limitations in the measurement of higher moments of current fluctuations with the emphasis on the third moment. We consider the effects of environmental noise added to the signal after amplifying, the effect of a Gaussian noise in the electromagnetic environment, and the averaging time.

A. Effect of amplifier noise

In typical measurement systems, the signal coming from the filters has to be amplified before it is mixed and detected. This amplification introduces additional noise $\delta I_a(t)$ in the output signal. In the following, we analyze the effect of such noise, assuming that it is Gaussian and uncorrelated with the fluctuations in the current coming from the sample. The second moment of such fluctuations is

$$S_a(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int d(t - t') \langle \delta I_a(t) \delta I_a(t') \rangle e^{i\omega(t-t')}, \qquad (49)$$

whereas the average and the other odd moments vanish. Moreover, one has to assume that this noise is intrinsically band-limited (due to finite RC-times, etc.), i.e.,

$$\langle \delta I_a(t) \delta I_a(t) \rangle = \frac{1}{\sqrt{2}} \int d\omega S_a(\omega) \equiv \frac{1}{\sqrt{2}} S_{\text{tot}}^a \qquad (50)$$

is finite. Due to this noise, the fluctuating signal before mixing is of the form

$$X(t) = \frac{1}{\sqrt{2\pi}} H_{BP}(t) * I(t) + \delta I_a(t). \qquad (51)$$

In the AC mixing scheme, the resulting signal after mixing is

$$\langle Y \rangle = \langle Y \rangle_0 + \frac{3}{2\pi} \{ H_{LP}^m * [f(t)[H_{BP}(t) * \langle I(t) \rangle] \langle \delta I_a(t)^2 \rangle] \}, \qquad (52)$$

where the first term has been calculated above and the second is due to excess noise. Such noise contributes only if the signal after band-pass filtering has a finite average and we apply $f(t) = \cos(\omega_0(t+t_0))$,

$$\langle Y \rangle = \langle Y \rangle_0 + \frac{3}{4\pi} S_{\text{tot}}^a I_1 \cos(\omega_0 t_0). \qquad (53)$$

For $f(t) = \cos(3\omega_0(t+t_0))$, the amplifier noise contribution vanishes.

In the AC statistics scheme, the amplifier noise does not contribute to the average signal. The second moment becomes

$$\langle (\delta Y)^2 \rangle = \langle (\delta Y)^2 \rangle_0 + \frac{1}{\sqrt{2\pi}} S_a(\omega_0) \omega_m, \qquad (54)$$

where the first term is that calculated above, and the second is due to amplifier noise within the band $\omega \in [\omega_0 - \omega_m, \omega_0 + \omega_m]$. As in the AC mixing scheme, the effect of amplifier noise is finite only for a nonvanishing average signal,
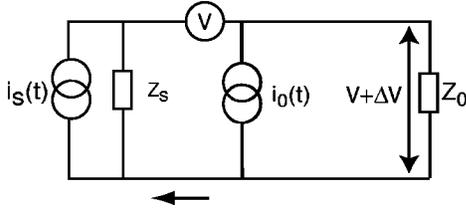
FIG. 3. Circuit studied for the effect of fluctuations in the electromagnetic environment on the third moment of the measured fluctuations. Circuit consists of the measured system with impedance Z_s and fluctuating current $i_s(t)$, coupled to its environment with impedance Z_0 and Gaussian current fluctuations $i_0(t)$. The arrow on the bottom of the figure indicates the chosen reference direction for the current.

$$\langle(\delta Y)^3\rangle = \langle(\delta Y)^3\rangle_0 + \frac{3}{4\pi}S_a(\omega_0)\omega_m\langle I_1\rangle\cos(\omega_0 t_0). \quad (55)$$

Apart from the noise in the amplifiers between the filtering and mixing, there may be other sources of noise. The most important noise source, that in the electromagnetic environment seen by the sample is treated below. Another source of noise, possibly relevant in the mixing schemes, is the noise added by the mixers. However, there the dependence of the signal on the phase t_0 can be used to distinguish the measured signal from the noise.

B. Linear system with environment

The effect of electromagnetic environment becomes the more important the higher moments one measures. For the second moment, the noise in the environment simply adds to that in the measured sample and scales the signal from the sample through the impedance ratio. Recently, Beenakker, Kindermann, and Nazarov[26] showed that the environment has a strong effect on the third moment, as this includes the effect of current noise in the environment biasing the sample. In the following, we generalize their approach to the case of nonstationary driving. We also address the methods for filtering in these devices and show how this yields the results of the general analysis in Sec. II.

To properly deal with the filtering, let us assume that the measured system can be separated in two parts (see Fig. 3): measured sample with impedance Z_s and a band-pass filter with impedance $Z_0(\omega)$. The impedances of the measuring lines, current amplifiers, etc. can be included in $Z_0(\omega)$. For simplicity, we assume that within the considered frequency interval, Z_s is frequency independent, and $Z_0(\omega)$ is given by

$$Z_0(\omega) = \begin{cases} Z_p, & \omega \in \pm[\omega_0 - \delta\omega, \omega_0 + \delta\omega], \\ Z_e, & \text{otherwise.} \end{cases} \quad (56)$$

Here, Z_e is either much larger (in the case when one would measure directly the current fluctuations) or much smaller (when converting the current fluctuations to voltage fluctuations) than the sample impedance. For simplicity, all impedances are chosen real. Moreover, assume that the current fluctuations in $Z_0(\omega)$ are Gaussian with the second moment given by

$$S_I^f(\omega,\omega') \equiv \langle\delta I_0(\omega)\delta I_0(\omega')\rangle = \frac{2k_B T_0}{Z_0(\omega)}\delta(\omega+\omega'). \quad (57)$$

Here T_0 characterizes the noise temperature of the environment and in general it may differ from the temperature of the sample.

The intrinsic current fluctuations $\delta I(\omega)$ in the two impedances cause voltage fluctuations $\Delta V(\omega)$ over them. As a result, the total current fluctuations at the two impedances are of the form

$$\Delta I_0(\omega) = \delta I_0(\omega) - \frac{\Delta V(\omega)}{Z_0(\omega)}, \quad (58)$$

$$\Delta I_s(\omega) = \delta I_s(\omega) + \frac{\Delta V(\omega)}{Z_s(\omega)}. \quad (59)$$

Assuming low enough frequencies (compared to the frequencies describing charge relaxation), the two currents should equal. This requirement leads to

$$\Delta V(\omega) = \frac{Z_s Z_0(\omega)}{Z(\omega)}[\delta I_0(\omega) - \delta I_s(\omega)], \quad (60)$$

$$\Delta I(\omega) = \frac{Z_s}{Z(\omega)}\delta I_s(\omega) + \frac{Z_0(\omega)}{Z(\omega)}\delta I_0(\omega). \quad (61)$$

Here we defined $Z(\omega)=Z_s+Z_0(\omega)$. The second moments of current fluctuations in the total system and voltage fluctuations over the impedances are given by[27]

$$S_I(\omega,\omega') \equiv \langle\Delta I(\omega)\Delta I(\omega')\rangle$$
$$= \frac{Z_s^2}{Z(\omega)Z(\omega')}S_I^s(\omega,\omega') + \frac{Z_0(\omega)Z_0(\omega')}{Z(\omega)Z(\omega')}S_I^f(\omega,\omega'), \quad (62)$$

$$S_V(\omega,\omega') \equiv \langle\Delta V(\omega)\Delta V(\omega')\rangle$$
$$= \frac{Z_s^2 Z_0(\omega)Z_0(\omega')}{Z(\omega)Z(\omega')}(S_I^s(\omega,\omega') + S_I^f(\omega,\omega')), \quad (63)$$

where $S_I^s(\omega,\omega') \equiv \langle\delta I_s(\omega)\delta I_s(\omega')\rangle$ measures the second moment of current fluctuations in the sample.

One can easily find that if the fluctuations are measured from the current, the optimal situation for noise measurement is $Z_p \ll Z_s$ and $Z_e \gg Z_s$ whereas for the measurement of voltage fluctuations, the maximum signal is obtained with $Z_p \approx Z_s$ and $Z_e \ll Z_s$.[28] In practice, when the signal power has to be compared to the resolution of the meters, the latter case is advantageous.

The major invention by Beenakker *et al.* was the fact that as the magnitude of current fluctuations in mesoscopic samples depends on the amplitude of driving, the voltage fluctuations in the environment tune these fluctuations, and thereby induce cross-correlations between the current fluctuations in the sample and in the environment. Following their work, we expand the second moment of fluctuations, $S_I^e(V)$ in the driving voltage near the average voltage,

$$\delta I_s(\omega)\delta I_s(\omega') = \int dt\, e^{i(\omega+\omega')t}(\delta I_s(t))^2$$

$$\approx \int dt\, e^{i(\omega+\omega')t}\left(S_I^s[\langle V(t)\rangle] + \Delta V(t)\right.$$

$$\left.\times \left.\frac{dS_I^s(V)}{dV}\right|_{v=\langle V(t)\rangle}\right). \qquad (64)$$

The first-order fluctuation term is enough to describe the environmental effect on the measured third moment. Now assume the driving voltage has the form given by Eq. (6) and expand the above two terms in Fourier harmonics,

$$S_I^s[\langle V(t)\rangle] = \sum_{n=o}^{\infty} S_n \cos(n\omega_0 t), \qquad (65)$$

$$\left.\frac{dS_I^s(V)}{dV}\right|_{V=\langle V(t)\rangle} = \sum_{n=0}^{\infty} D_n \cos(n\omega_0 t). \qquad (66)$$

For example, in a linear system in the limit $eV \gg k_B T$, driving with $V(t) = V_3 \cos(3\omega_0 t)$ yields

$$D_1 = 0, \quad D_3 = \frac{4F_2 e}{\pi Z_s}, \qquad (67)$$

where F_2 is the Fano factor for the second moment.

Combining, we get

$$\delta I_s(\omega)\delta I_s(\omega') \approx \frac{1}{2}\sum_n \left[S_n(\delta(\omega+\omega'+n\omega_0) \right.$$

$$+ \delta(\omega+\omega'-n\omega_0)) + D_n(\Delta V(\omega+\omega'+n\omega_0)$$

$$\left. + \Delta V(\omega+\omega'-n\omega_0)) \right]. \qquad (68)$$

The third moment of the current fluctuations $\Delta I(\omega)$ contains three parts: first coming from the "intrinsic" third moment of the measured sample, K_I^s, second from the cross-correlations between the sample and its environment, and the third from the current fluctuations of the sample driving the sample itself[26]

$$K_I(\omega,\omega',\omega'') \equiv \langle \Delta I(\omega)\Delta I(\omega')\Delta I(\omega'')\rangle$$

$$= \left(\frac{Z_s^3}{Z(\omega)Z(\omega')Z(\omega'')}\right)\left[K_I^S(\omega,\omega',\omega'') + k_B T_0 \right.$$

$$\times \left(\frac{Z_0(\omega)}{Z(\omega)} + \frac{Z_0(\omega')}{Z(\omega')} + \frac{Z_0(\omega'')}{Z(\omega'')}\right)$$

$$\times \sum_{n,\alpha=\pm1} D_n \delta(\omega_s + n\alpha\omega_0) - \frac{Z_s}{4}$$

$$\times \sum_{n,m\alpha,\beta=\pm1} D_n S_m \left(\frac{Z_0(\omega+m\beta\omega_0)}{Z(\omega+m\beta\omega_0)}\right.$$

$$\left.+ \frac{Z_0(\omega'+m\beta\omega_0)}{Z(\omega'+m\beta\omega_0)} + \frac{Z_0(\omega''+m\beta\omega_0)}{Z(\omega''+m\beta\omega_0)}\right)$$

$$\left. \times \delta(\omega_s + (n\alpha+m\beta)\omega_0) \right], \qquad (69)$$

where $\omega_s = \omega+\omega'+\omega''$. This correlator is essential if one is able to directly measure the current fluctuations. In this case, specifying $Z_e \gg Z_s$, the fluctuations outside the band ω,ω', $\omega'' \in \pm[\omega_0-\delta\omega, \omega_0+\delta\omega]$ can be neglected and within this band,

$$K_I(\omega,\omega',\omega'') = \frac{Z_s^3}{(Z_s+Z_p)^3}\left[\left(\frac{K_1}{2} + \frac{3D_1}{2}\frac{Z_p}{(Z_s+Z_p)^2}\right)\right.$$

$$\times (2k_B T_0 - Z_s S_0) - \frac{3Z_s}{4}\sum_{m\neq0,n\pm m=1} D_n S_m\right)$$

$$\times \delta(\omega_s \pm \omega_0) + \left(\frac{K_3}{2} + \frac{3D_3}{2}\frac{Z_p}{(Z_s+Z_p)^2}\right.$$

$$\times (2k_B T_0 - Z_s S_0) - \frac{3Z_s}{4}\sum_{m\neq0,n\pm m=3} D_n S_m\right)$$

$$\left.\times \delta(\omega_s \pm 3\omega_0)\right]. \qquad (70)$$

Other Fourier components of K and D vanish due to the filtering provided by Z_e.

Another important observable is the third moment of voltage fluctuations over the impedance Z_0. This is given by

$$K_V(\omega,\omega',\omega'') = \frac{Z_s^3 Z_0(\omega)Z_0(\omega')Z_0(\omega'')}{Z(\omega)Z(\omega')Z(\omega'')}\left[-K_I^s + Z_s k_B T_0\right.$$

$$\times \left(\frac{1}{Z(\omega)} + \frac{1}{Z(\omega')} + \frac{1}{Z(\omega'')}\right)$$

$$\times \sum_{n,\alpha=\pm1} D_n \delta(\omega_s + n\alpha\omega_0)$$

$$+ \frac{Z_s}{4}\sum_{n,m\alpha,\beta=\pm1} D_n S_m \left(\frac{Z_0(\omega+m\beta\omega_0)}{Z(\omega+m\beta\omega_0)}\right.$$

$$\left.+ \frac{Z_0(\omega'+m\beta\omega_0)}{Z(\omega'+m\beta\omega_0)} + \frac{Z_0(\omega''+m\beta\omega_0)}{Z(\omega''+m\beta\omega_0)}\right)$$

$$\left.\times \delta(\omega_s + (n\alpha+m\beta)\omega_0)\right]. \qquad (71)$$

Now the pass-band may be defined through $Z_p \approx Z_s$ and the stop-band by $Z_e \ll Z_s$. Assume for simplicity that $Z_p = Z_s$. Then, if all the frequencies are within the pass-band, one obtains

$$K_V(\omega,\omega',\omega'') = \frac{Z_s^3}{8}\left[-\frac{K_1}{2}\delta(\omega_s \pm \omega_0) - \frac{K_3}{2}\delta(\omega_s \pm 3\omega_0)\right.$$

$$+ \frac{3}{4}(Z_s S_0 + 2k_B T_0)[D_1\delta(\omega_s \pm \omega_0)$$

$$\left.+ D_3\delta(\omega_s \pm 3\omega_0)]\right]. \qquad (72)$$

In the stop-band, K_V is negligible.[27]

As an example, consider a linear system with Fano factors F_2 and F_3 (for the second and third moments) in an impedance-matched environment $Z_p = Z_s$ and driven with

$V(t) = V_3 \cos(3\omega_0 t)$, $eV_3 \gg k_B T$. The third moment of voltage fluctuations over the sample within the pass-band is

$$K_V = Z_s^3 \left[-\frac{e^2 F_3 I_3}{16} + \frac{3}{4\pi^2} F_2^2 e^2 I_3 + \frac{3}{\pi} F_2 e \frac{k_B T_0}{Z_s} \right] \delta(\omega_s \pm 3\omega_0). \tag{73}$$

This equation illustrates how the measured third moment consists of the "real signal" (dependent on F_3) and the parts coming from the sample noise driven by its own fluctuations (second term) and those in the environment (third term).

In their paper,[21] Reulet *et al.* showed that the environmental effect can be accurately described by this theory after including the finite propagation time τ in the coaxial cable between the filter and the sample (see Fig. 3). Their analysis is analogous in the case of slow driving, and indicates that a long cable can be used to decrease the effective noise temperature T_0 of the environment.

C. Averaging time

In Ref. 21, the required time for signal averaging was several hours for each plot. Naturally, for any reasonable measurement, this time should not be much longer. To get an estimate of the required measurement time with the cyclostationary driving, we consider the AC statistics scheme (the requirements are analogous for the first scheme, only the prefactors of the expressions may slightly differ). There, one obtains a single output value in a time determined by the data integration time $t_{meas} = 2\pi/\omega_m$. Assume one has measured n values of current. The squared deviation between the measured value of the unbiased estimator k_3 for the third moment of the data and the true third moment can be estimated through the variance of k_3.[2] In the limit $n \gg 1$ it is given by

$$\text{var}(k_3) = \frac{1}{n}(9\mu_2^3 - \mu_3^2 - 6\mu_2\mu_4 + \mu_6), \tag{74}$$

where μ_i are the ith central moments of the true current distribution. Now let us require that the relative error is smaller than some percentage p of μ_3,

$$\text{var}(k_3) < p^2 \mu_3^2. \tag{75}$$

This leads to the requirement

$$n > \frac{1}{p^2}\left(\frac{9\mu_2^3}{\mu_3^2} - 1 - 6\frac{\mu_2\mu_4}{\mu_3^2} + \frac{\mu_6}{\mu_3^2} \right). \tag{76}$$

From this example, it is thus clear that any additional noise from the measuring setup increases the measurement time through the increase of the even moments.

In the typical limit where the current I through the sample exceeds e times the bandwidth ω_d (e.g., for $I = 100$ nA, $I/e = 620$ GHz), it is safe to neglect the cumulants of higher order than three, i.e., the measured data is almost Gaussian. In this case the above requirement reduces to

$$n > \frac{6}{p^2}\frac{\mu_2^3}{\mu_3^2} = \frac{32\pi}{3p^2\omega_d}\frac{[2S_0 + S_2\cos(2\omega_0 t_0)]^3}{[3K_1\cos(\omega_0 t_0) + K_3\cos(3\omega_0 t_0)]^2}, \tag{77}$$

where the last form was obtained using Eqs. (29) and (30).

To get an idea of the required measuring times, consider a linear sample with resistance R_S, driven with the voltage $V_3\cos(3\omega_0 t)$ in the shot-noise limit $eV_3 \gg k_B T$. Assume that the second moment can be described by a combination of the shot noise over the sample and the amplifier noise, the latter described by the noise temperature T_0 and resistance R_0. In the case $R_S \gg R_0$, the two should be matched via a matching circuit. This can be done, for example, by placing a capacitance C in parallel to the sample and an inductance L in series with the two. In the case of optimal matching, $L = \sqrt{R_0 R_S}/\omega_0$ and $C = (\sqrt{R_0 R_S}\omega_0)^{-1}$, the total dc voltage noise over the amplifier is given by $S_0 = 2R_0 R_S(k_B R_t T_0 + eR_S F_2 V_3/\pi)/(R_t(R_0 + 4R_S))$, where $R_t = R_0 + R_S$. The third cumulant of the voltage noise over the amplifier, oscillating with the frequency $3\omega_0$, is then given by $K_3 = -e^2 R_0^2(R_0 - 12R_S)R_S^3 F_3 V_3/(R_t(R_0 + 4R_S)^3)$. For simplicity, we only take the direct contribution to the third cumulant, as those induced by the environment (see IV B) are typically of the same order of magnitude. Thus we get the required number of measured values,

$$n > \frac{1024(R_0 + 4R_S)^3(k_B\pi R_t T_0 + eF_2 R_S V_3)^3}{3f_d p^2\pi^3 e^4 R_0(R_0 - 12R_S)^2 R_S^3 R_t F_3^2 V_3^2}. \tag{78}$$

Here we denoted the bandwidth $f_d = \omega_d/(2\pi)$ by the real frequency instead of the angular frequency. The measurement time is $T = n/f_m$, where $f_m = \omega_m/(2\pi)$ is the sampling frequency, assumed in the following to equal f_d. It has a minimum when the shot noise power from the sample equals the amplifier noise, i.e., when $eV_3 = 2k_B TR_t/(\pi F_2 R_S)$. For this voltage, the requirement for the measurement time is $T > T_m$, where

$$T_m = \frac{2304 F_2^2 k_B(R_0 + 4R_S)^3 T_0}{F_3^2 f_d^2 p^2\pi^2 e^2 R_0(R_0 - 12R_S)^2 R_S}. \tag{79}$$

In the limit $R_S \gg R_0$, this becomes independent of R_S,

$$T_m \to \frac{1024 k_B T_0}{f_d^2 p^2\pi^2 e^2 R_0}. \tag{80}$$

As an example, in the case of a tunnel junction, $F_2 = F_3 = 1$, for $f_d = 100$ MHz, $R_0 = 50$ Ω, $p = 0.05$, $T_0 = 1$ K, we obtain $n \gtrsim 4.5 \times 10^9$ measurement points and the measurement time $T_m = 45$ s. Note that the bandwidth of the matching is proportional to $\sqrt{R_0/R_S}$. Thus, if this is the limiting factor to f_d (but not f_m), the averaging time increases as $\sqrt{R_S/R_0}$ as the resistance of the sample is increased.

A similar estimate for the second moment yields a sample size (in the limit $n \gg 1$)

$$n > \frac{1}{p^2\mu_2^2}(\mu_4 - \mu_2^2) \approx \frac{2}{p^2} \tag{81}$$

independent of the current or the bands (this is analogous to the Dicke radiometer formula[29]). Hence, for $p = 0.01$, we would need some tens of thousand samples, and the required measuring time with MHz sampling would be of the order of milliseconds.

TABLE I. Values of g_n^l for different moments n and different harmonics l.

n	1	2	2	3	3	4	4	4	5	5	5	6	6	6	6
l	1	0	2	1	3	0	2	4	1	3	5	0	2	4	6
g_n^l	1	1	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{5}{8}$	$\frac{5}{16}$	$\frac{1}{16}$	$\frac{5}{8}$	$\frac{15}{32}$	$\frac{3}{16}$	$\frac{1}{32}$

V. SUMMARY

In this paper, we detail methods to measure the moments of current fluctuations by employing a nonstationary driving signal. Such driving overcomes the requirement for the wide-band measurements and gives a way to confirm the measurement results through the variation of the phase between the signal and the mixing. However, cyclostationary driving makes rise to higher harmonics of the signal, which have to be taken care of through a proper Fourier analysis. In Sec. III, we discuss a few examples of such an analysis.

Section IV shows that the measurement of the third moment always involves the effect caused by the measuring setup. The averaging times discussed at the end of the same section indicate that this type of "classical" measurements are practical for the measurement of the third and perhaps still for the fourth cumulant, but higher cumulants seem to be out of the time constants set for any reasonable project. Therefore, schemes where the detector is a mesoscopic system and placed near the sample,[19,20,30] overcoming the low-bandwidth restrictions, are preferable.

Let us outline a practical scheme for the detection of the third moment/cumulant of current fluctuations with cyclostationary driving. Assume one measures a sample with (possibly voltage-dependent) impedance Z_s.

(1) Construct the matching circuit that approximately matches the input impedance of the amplifier to that of the sample, and filters the signal from the narrow band $\omega \in \pm[\omega_0 - \delta\omega, \omega_0 + \delta\omega]$. Note that the measurement outcome will depend on the details of the matching, as discussed in Sec. IV B.

(2) Drive the sample with voltage $V(t) = V_0 + V_3\cos(3\omega_0 t)$ where either $V_0 = 0$ or $V_0 \gg V_3$ ("differential measurement," see Sec. III).

(3) Either mix the signal twice with itself and once with $f(t) = \cos(3\omega_0(t+t_0))$ (Sec. II A 2) or only with $f(t) = \cos(\omega_0(t+t_0))$ (Sec. II B). Average the signal (mixing scheme) or its third moment (statistics scheme) for time exceeding that indicated in Sec. IV C. The outcome is given by Eq. (24) or Eq. (30), where K_3 is taken from the $\cos(3\omega_0 t)$ component of the Fourier series of $K(V(t))$ as in Secs. III A and III B.

APPENDIX A: MOMENTS IN THE AC STATISTICS SCHEME

The contribution of the lth harmonic of the nth moment, M_n^l, to the nth measured moment in the AC statistics scheme can be written as

$$\langle(\delta Y)^n\rangle_l = \frac{M_n^l}{(2\pi)^n}\int dt_1\ldots dt_n H_{LP}^m(t_1)\ldots H_{LP}^m(t_n)f(t_1)\ldots f(t_n)$$
$$\times \int dt' H_{BP}(t_1-t')\ldots H_{BP}(t_n-t')\cos(l\omega_0 t'). \tag{A1}$$

The convolution between the H_{BP}-functions filters out most of the harmonics l. It only leaves $l = 1, 3, \ldots, n$ in the case when n is odd and $l = 0, 2, \ldots, n$ when n is even.

Now, use the fact that $H_{BP}(t) = 2\cos(\omega_0 t)H_{LP}^\delta(t)$, where H_{LP}^δ is a low-pass filter with band $\omega \in [-\delta\omega, \delta\omega]$. One obtains

$$\langle(\delta Y)^n\rangle_l = M_n^l\left(\frac{1}{\pi}\right)^n \int dt_1\ldots dt_n dt' H_{LP}^m(t_1)\ldots H_{LP}^m(t_n)$$
$$\times H_{LP}^\delta(t_1-t')\ldots H_{LP}^\delta(t_n-t')G(t_1,\ldots,t_n,t'), \tag{A2}$$

where

$$G(t_1,\ldots,t_n,t') = f(t_1)\ldots f(t_n)\cos(\omega_0(t_1-t'))\ldots$$
$$\times\cos(\omega_0(t_n-t'))\cos(l\omega_0 t'). \tag{A3}$$

If $\omega_m, \delta\omega < \omega_0/n$, only the time-independent part of $G(t_1,\ldots,t_n,t')$ survives the remaining filtering. This can be computed from

$$G_{DC}^{n,l} \equiv \left(\frac{\omega_0}{2\pi}\right)^{n+1}\int_0^{2\pi/\omega_0} dt_1\ldots dt_n dt' G(t_1,\ldots,t_n,t'). \tag{A4}$$

However, $G_{DC}^{n,l}$ may still depend on the relative phase t_0 between the driving and the mixed function $f(t) = \cos(\omega_0(t+t_0))$. Equivalently to above, $G_{DC}^{n,l}$ is only finite if l is one of the values that survives band-pass filtering. Moreover, it is straightforward to show that $G_{DC}^{n,l}(t_0) = g_n^l\cos(l\omega_0 t_0)/2^{n+1}$. The number g_n^l for some of the lowest (nontrivial) values of n,l is tabulated in Table I.

The remaining part of the integral in Eq. (A2) is now a combination of low-pass filters:

$$\langle(\delta Y)^n\rangle_l = M_n^l g_n^l \cos(l\omega_0 t_0)\frac{1}{2}\left(\frac{1}{2\pi}\right)^n \int dt_1 dt_2\ldots dt_n dt'$$
$$\times H_{LP}^m(t_1)\ldots H_{LP}^m(t_n)H_{LP}^\delta(t_1-t')\ldots H_{LP}^\delta(t_n-t'). \tag{A5}$$

Singling out one of the t_i integrals, it is straightforward to show that

$$\int dt_i H_{\text{LP}}^m(t_i) H_{LP}^\delta(t_i - t') = \sqrt{2\pi} F_{t'}^{-1}\left[\widetilde{H}_{LP}^m(\omega)\widetilde{H}_{LP}^\delta(\omega)\right]$$

$$= \sqrt{2\pi} H_{LP}^d(t'), \qquad (A6)$$

where $H_{LP}^d(t')$ is a low-pass filter with bandwidth $\omega_d = \min(\omega_m, \delta\omega)$ ($\omega \in [-\omega_d, \omega_d]$). Thus we get

$$\langle(\delta Y)^n\rangle_l = \frac{1}{2}\left(\frac{1}{2\pi}\right)^{n/2} M_n^l g_n^l \cos(l\omega_0 t_0)\mathcal{H}_n(\omega_d). \qquad (A7)$$

Here

$$\mathcal{H}_n(\omega_d) = \int dt' H_{LP}^d(t')^n$$

$$= \frac{1}{(2\pi)^{n/2-1}} \times \int d\omega_1 \ldots d\omega_{n-1} H_{LP}^d(\omega_1)\ldots H_{LP}^d(\omega_{n-1})$$

$$\times H_{LP}^d(\omega_1 + \cdots \omega_{n-1})$$

$$\equiv \frac{\alpha_n \omega_d^{n-1}}{(2\pi)^{n/2-1}}. \qquad (A8)$$

and α_n is a number whose values for lowest n are

$$\begin{array}{ll} \alpha_1 = 1, & \alpha_2 = 2, \\ \alpha_3 = 3, & \alpha_4 = 16/3, \\ \alpha_5 = 115/12, & \alpha_6 = 88/5, \\ \alpha_7 = 17\,407/360, & \alpha_8 = 53\,752/315, \\ \alpha_9 = 18\,063\,361/40\,320, & \alpha_{10} = 1440. \end{array} \qquad (A9)$$

Finally the contribution of the lth harmonic of the nth moment on the nth moment of the measured quantity is given by

$$\langle(\delta Y)^n\rangle = \frac{1}{2^n\pi^{n-1}} g_n^l \alpha_n \omega_d^{n-1} M_n^l \cos(l\omega_0 t_0). \qquad (A10)$$

The lowest six moments are listed in Eqs. (28)–(33).

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